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SOLUTIONS IN A NON-ANTAGONISTIC POSITIONAL DIFFERENTIAL GAME†

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A new approach to constructing solutions in a two-person non-antagonistic positional differential game (NPDG) is proposed. The approach is based on the use of the principle of non-degradation of guaranteed results [1, 2] of the players along trajectories which generate solutions on the use of the rule of extremal shift [1, 2] in the direction of Pareto-optimal states, that are best for one and for the other players and, finally, on the use of equilibrium solutions in special auxiliary bimatrix games. Splitting of the set of positions in NPDG is proposed. Non-antagonistic positional differential games are considered with different types of player behaviour. This paper is an extension of the investigation described in [3, 4].

1. SOME RESULTS FROM THE THEORY OF NPDG [3]

Suppose the dynamics of the controlled system is described by the equation

$$\dot{x} = f(t, x, u, v), \quad x(t_0) = x_0 \tag{1.1}$$

where $x \in R^n$ is a phase vector and the controls $u \in P \in \text{comp}\, R^p$ and $v \in Q \in \text{comp}\, R^q$ obey the first and second players, respectively. In the space of the variables t, x a compact g is specified, the projection of which on the t axis is equal to the specified section $[t_0, \vartheta]$, where ϑ is the fixed instant when the process is completed; here all the trajectories of system (1.1), beginning at an arbitrary position $(t_*, x_*) \in G$, remain within G for all $t \in (t_*, \vartheta]$. We will assume that the function f: $G \times P \times Q \mapsto R^n$ is continuous over the set of arguments, and satisfies the Lipschitz condition with respect to x, and also the continuability condition of all trajectories of system (1.1) in the section $[t_*, \vartheta]$.

We will also assume, for simplicity, that the vectorgram of system (1.1)

$$F(t, x) = \{s \in R^n : s = f(t, x, u, v), u \in P, v \in Q\} \tag{1.2}$$

is convex at each point $(t, x) \in G$. This assumption ensures that the set of continuability generated by the measurable controls is closed with respect to uniform convergence.

We will assume that the player with number i tries to maximize the terminal performance figure

$$I_i = \sigma_i(x(\vartheta)), \quad i = 1, 2 \tag{1.3}$$

where σ_i: $R^n \mapsto R^1$ are continuous functions.

Both players have complete information on the current position $(t, x(t))$ of the game. The strategy of the players, and also the motions they generate, are formalized in the NPDG just as in the theory of antagonistic positional differential games (APDG) [1, 2], with the exception of technical details. The strategy of the first player is identified with the pair $U + \{u(t, x, \varepsilon), \beta_1(\varepsilon)\}$, while the strategy of the second player is identified with the pair $V + \{v(t, x, \varepsilon), \beta_2(\varepsilon)\}$. Here u: $G \times (0, \infty) \mapsto P$ and v: $G \times (0, \infty) \mapsto Q$ are arbitrary functions, while β_i:$(0, \infty) \mapsto (0, \infty)$ are continuous monotonic functions which satisfy the condition $\beta_i \to 0$ as $\varepsilon \to 0$. For a fixed value of ε the quantity $\beta_i(\varepsilon)$ specifies the upper limit on the subdivisions of the section $[t_0, \vartheta]$ used by player i when constructing the Euler broken lines. The parameter ε is called the accuracy parameter [3]. The pair of strategies (U, V) generates two types of motion: approximation motions (Euler broken lines) $x[\cdot, t_0, x_0, U, \varepsilon_1, \Delta_1, V, \varepsilon_2, \Delta_2]$ and ideal (limit) motions $x[\cdot, t_0, x_0, U, V)$. The approximation motions are defined for fixed subdivisions by the players of the section $[t_0, \vartheta] : \Delta_1 = \{t_i^1\}$ and $\Delta_2 = \{t_i^2\}$ and for fixed values of the accuracy parameters of the players ε_1 and ε_2. The pair of strategies (U, V) generates a non-empty compact set of limit motions (in the metric of

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the space $C[t_0, \vartheta]$). Without loss of generality we will henceforth consider only those pairs of strategies (U, V) which generate unique limit motions.

The problem of determining the idea of the solution is one of the central problems of the theory of NPDG. In order that the idea of the solution should be adequate to the problem, it is necessary, when determining the solution, to take into account all the limitations which exist in the game. This was done in [3] by introducing the following additional elements of the game:

1. a description of the set of players who may deviate from the solution and the set of permitted instants of deviation for each player;
2. a description of the behaviour of the player-pursuer after evasion of the player–evader;
3. additional limitations on the permissible sets of strategies;
4. a description of the sequence in which the players chose sets of strategies as the solution of the game.

Below we consider the following additional elements of the game:

1. each player may singly deviate from the solution at any instant of time $t \in [t_0, \vartheta)$;
2. a player–pursuer, after evading the player–evader may choose any suitable strategy including the worst one for the evader;
3. we will consider two versions: the first version, when there are no additional limitations; we will denote the game in this version by MPDG-1; the second version, when the condition of "care" of the players is imposed, which consists of the fact that the guaranteed payoffs of the players are not produced along the trajectory; the game in this version will be denoted by MPDG-2;
4. the players choose sets of strategies as the solutions simultaneously.

Consider auxiliary APDG, denoted henceforth by Γ_1 and Γ_2. In game Γ_i the ith player maximizes the performance figure $\sigma_i(x(\vartheta))$ (1.3), while the $(3-i)$th player does the opposite. We will assume that the function $f(t, x, u, v)$ on the right-hand side of (1.1) satisfies the condition

$$\max_{u\in P}\min_{v\in Q} s^T f(t,x,u,v) = \min_{v\in Q}\max_{u\in P} s^T f(t,x,u,v) \tag{1.4}$$

for any choice of $s \in R^n$ and $(t, x) \in G$. Equation (1.4) was called the saddle-point condition in the small game in [1]. When this condition is satisfied the games Γ_1 and Γ_2 have universal saddle points in the class of pure positional strategies

$$\{u^{(i)}(t,x,\varepsilon), v^{(i)}(t,x,\varepsilon)\}, \quad i=1,2 \tag{1.5}$$

and the value functions

$$\gamma_1(t,x), \quad \gamma_2(t,x) \tag{1.6}$$

The universality of strategies (1.5) means that they are suitable not only for the initial position (t_0, x_0) but also for any position $(t, x) \in G$ taken as the initial position. The quantity $\gamma_i(t, x)$ for the position (t, x) is the guaranteed payoff of the ith player for any actions of his or her partner.

Definition 1. The set of strategies (U, V) will be said to be permissible in the NPDG if no player deviates disadvantageously from this set at any of the permitted instants of deviation.

In definition 1 the deviation of a player is assumed to be an advantage if, as a result of the deviation, he obtains a more guaranteed payoff than in a game without deviations.

It is obvious that if a set of strategies is not permissible, it cannot be chosen as a solution of the game. We will denote by D the non-empty set of permissible sets of strategies. Further, we will denote by D^* the subset of the set D, consisting of permissible sets of strategies which satisfy the additional constraints 3. Clearly for the game NPDG-1 we have $D^* = D$, while for game NPDG-2 the subset D^* contains only those permissible sets of strategies which generate trajectories with monotonic non-decreasing functions $\gamma_1(t, x(t))$ and $\gamma_2(t, x(t))$ along them.

Definition 2. Any element of the set D^* that is not Pareto-improvable with respect to the performance figures I_1 and I_2 given by (1.3) will be called a solution of the NPDG.

We will call the solution introduced the P-solution. In general, there are infinitely many P-solutions. Among these we can distinguish the solution that is best for the first player (we will call it the H_1-solution) and the solution that is best for the second player (we will call it the H_2-solution). The H_1-solution and H_2-solution cannot be the same.

In general, the P-solutions and, in particular, the H_1-solution and H_2-solution have the following structure

$$u(t,x,\varepsilon)=\begin{cases} u^*(t) & \text{for } \|x-x^*(t)\|\leqslant \varepsilon\varphi(t) \\ u^2(t,x,\varepsilon) & \text{for } \|x-x^*(t)\|>\varepsilon\varphi(t) \end{cases}$$

$$\upsilon(t,x,\varepsilon)=\begin{cases} \upsilon^*(t) & \text{for } \|x-x^*(t)\|\leqslant \varepsilon\varphi(t) \\ \upsilon^2(t,x,\varepsilon) & \text{for } \|x-x^*(t)\|>\varepsilon\varphi(t) \end{cases} \quad (1.7)$$

where $\upsilon^1(\cdot), u^2(\cdot)$ are defined in (1.5), the functions $u^*(t)$ and $\upsilon^*(t)$ are solutions of certain non-standard problems of programmed optimal control, and the functions $\varphi(t)$, $\beta_1(\varepsilon)$ and $\beta_2(\varepsilon)$ are consistent with one another.

2. SUBDIVISION OF THE SET OF POSITIONS IN NPDG

We introduce the following definitions.

Definition 3. We will call the position $(t_*, x_*) \in G$ a non-antagonistic position (an NA-position) if a trajectory of system (1.1) $x(t)$, $t_* \leqslant t \leqslant \vartheta$, $x(t_*) = x_*$ exists such that the functions $\gamma_i(t, x(t))$ ($i = 1, 2$) (1.6) do not decrease at $[t_*, \vartheta]$ and the following strict inequality is satisfied at least for one j

$$\gamma_j(\vartheta, x(\vartheta)) > \gamma_j(t_*, x_*) \quad (2.1)$$

Definition 3 means that from the NA-positions motion is possible along the trajectories along which guaranteed payoffs of the players do not decrease, and at least for one player this payoff at the end of the trajectory is strictly greater than at the initial NA-position. These trajectories will be called NA-trajectories. For an NA-position there is, generally speaking, infinitely many NA-trajectories.

Definition 4. The position $(t_*, x_*) \in G$ will be called a locally antagonistic position (an LA-position), if it is not a non-antagonistic position and, moreover, a trajectory of system (1.1) $x(t)$, $t_* \leqslant t \leqslant \vartheta$, $x(t_*) = x_*$ exists along which the following inequalities are satisfied

$$\gamma_i(\vartheta, x(\vartheta)) \geqslant \gamma_i(t, x(t))), \quad t_* \leqslant t \leqslant \vartheta, \quad i = 1, 2 \quad (2.2)$$

where at least one of inequalities (2.2) in strict for $t = t_*$.

Trajectories from Definition 4 will be called LA-trajectories. For an LA-position there are, generally speaking, infinitely many LA-trajectories. Inequalities (2.2) denote that the point $t = t_*$ is a point of maximum of both functions $\gamma_1(t, x)$ and $\gamma_2(t, x)$, calculated along the LA-trajectory. Hence, at the end of the LA-trajectory the payoff of both players will not be less than in the initial LA-position, and this payoff will be strictly greater for at least one player.

Definition 5. We will call the position $(t_*, x_*) \in G$ a globally antagonistic position (a GA-position) if for any trajectory of system (1.1) $x(t)$, $t_* \leqslant t \leqslant \vartheta$, $x(t_*) = x_*$ either we have the equalities

$$\gamma_i(\vartheta, x(\vartheta)) = \gamma_i(t_*, x_*), \quad i = 1, 2 \quad (2.3)$$

or at least for one j the following inequality is satisfied

$$\gamma_i(\vartheta, x(\vartheta)) < \gamma_i(\tau, x(\tau)) \quad (2.4)$$

for any $\tau \in [t_*, \vartheta)$.

Obviously, for a GA-position there is always at least one trajectory along which

$$\gamma_i(t, x(t)) \equiv \gamma_i(t_*, x_*), \quad t_* \leqslant t \leqslant \vartheta, \quad i = 1, 2 \quad (2.5)$$

This trajectory will be called a GA-trajectory. In general there can be infinitely many GA-trajectories. They are all on surfaces of the level of the functions $\gamma_1(t, x)$ and $\gamma_2(t, x)$ simultaneously. We note in passing that in GA-positions the players achieve Pareto-optimal payoffs.

Trajectories which satisfy inequality (2.4) are unsuitable for the jth player, and therefore cannot be realized in the game.

It is easy to see that for LA-positions, in addition to LA-trajectories there are also GA-trajectories while for NA-positions, in addition to NA-trajectories, there are LA-trajectories and GA-trajectories.

Hence, we obtain a subdivision of the set G of all positions in the NPDG into three subsets: subset G_1 of all NA-positions, subset G_2 of all LA-positions and subset G_3 of all GA-positions.

Theorem 1. If in the game NPDG-1 the initial position is an NA-position, then the NP-solution of the game generates either an NA-trajectory or an LA-trajectory. If the initial position is an LA-position (a GA-position), the P-solution of the game generates an LA-trajectory (a GA-trajectory).

Theorem 2. If in the NPDG-2 game the initial position is an NA-position (an LA-position or a GA-position), the P-solution of the game generates a NA-trajectory (a GA-trajectory).

The proof of Theorems 1 and 2 rests on Theorems 1.4 and 1.9 of [3] and on the above definitions.

We will consider two examples.

Example 1. The dynamics of the system is described by the equation

$$\dot{x} = (\vartheta - t)(u + v), \quad x, u, v \in R^2, \quad \|u\| \leq 1, \|v\| \leq 1 \tag{2.6}$$

and the performance figures of the players (1.3) have the form

$$\sigma_i(x(\vartheta)) = -\|x(\vartheta) - a^{(i)}\|, \quad i = 1, 2 \tag{2.7}$$

where $a^{(i)}$ are specified points in R^2.

The game problem of the plane motion of a point mass with two target points [3, Section 1.13, Example 3] reduces to this problem.

It can be shown that in this game the set G_3 of GA-positions consists of all those points $(t, x) \in G$ for which x belongs to the section connecting the target points $a^{(1)}$ and $a^{(2)}$. The set G_1 of NA-positions is made up of all the remaining positions $(t, x) \in G$. The set G_2 of LA-positions is an empty set in this example.

Example 2. The dynamics of the system is described by the equation

$$\dot{x} = u + v, \quad x, u, v \in R^2, \quad \|u\| \leq 1, \quad \|v\| \leq 1 \tag{2.8}$$

while the performance figures of the players (1.3) have the form

$$\sigma_1(x(\vartheta)) = -\|x(\vartheta) - a^{(1)}\|, \quad \sigma_1(x(\vartheta)) = \sqrt{3}\,|x_1(\vartheta)| - x_2(\vartheta) \tag{2.9}$$

This game problem was also considered previously [3, Section 1.13, Example 1] and it was established that the value functions (1.6) have the form

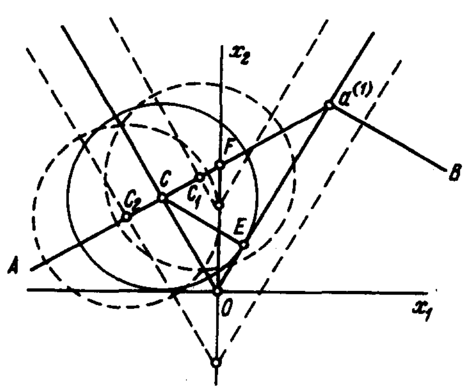

Fig. 1.

$$\gamma_1(t,x)=-\|x-a^{(1)}\|,\quad \gamma_2(t,x)=\sqrt{3}\,|x_1|-x_2 \tag{2.10}$$

We put $a^{(1)} = (1, \sqrt{3})$ (see Fig. 1). The level line of the function $\gamma_2(t, x)$ (2.10), passing through the point $a^{(1)}$, is a combination of the rays OC and $Oa^{(1)}$. We draw $Ca^{(1)}\perp OC$ and $CE\perp Oa^{(1)}$. We have $|CE| = \sqrt{(3)}/2$. Denoting the section of the set G by the hyperplane t = const by G^t, we will further confine ourselves to describing the subdivision of the set G^t into the subsets G_1^t, G_2^t, G_3^t with $t = \vartheta - \sqrt{(3)}/4$.

For the point C, as for the initial point and for the initial instant of time $t = \vartheta - \sqrt{(3)}/4$, the set of accessibility for system (2.8) at the instant ϑ will be a circle of radius $\sqrt{(3)}/2$, to the boundary of which the straight line $OA^{(1)}$ is tangent at the point E (see Fig. 1). Here the section CE specifies a unique LA-trajectory, and there is no NA-trajectory for this position. This means that the point C belongs to the set G_2^t when $t = \vartheta - \sqrt{(3)}/4$.

Similar discussions show that the point C_1 (like other points of CF, with the exception of the point F) belongs to the set G_2^t of LA-positions with $t = \vartheta - \sqrt{(3)}/4$. At the same time, the point C_2 (like other points of the ray CA, with the exception of the point C) will belong to the set G_3^t of GA-positions; the same can also be said of points of the ray $a^{(1)}B$, including the point $a^{(1)}$. All the remaining positions of G^t belong to the set G_1^t of NA-positions with $t = \vartheta - \sqrt{(3)}/4$.

3. A PROCEDURE OF SUCCESSIVE CONTRACTION OF THE SET OF P-SOLUTIONS

Consider the NPDG-2 game. Suppose the dynamics of the game are described by an equation with "separated" right-hand sides, i.e.

$$\dot{x}=f(t,x,u)+g(t,x,v),\quad x(t_0)=x_0 \tag{3.1}$$

where constraints similar to the constraints imposed on the function f in Section 1 are imposed on the functions f and g.

We define the function $\rho_1^0:G\mapsto R^1$ and $\rho_2^0:G\mapsto R^1$ assuming that $\rho_i^0(t,x)$ is the value of the functional (1.3) of player i in the H_i-solution if the position (t, x) is taken as the initial position.

We will denote by S^0 the set of payoff vectors of the players (I_1, I_2) (1.3), which are achieved on the P-solutions of the game.

We will now describe a procedure $L(S^0, \rho_1^0(\cdot), \rho_2^0(\cdot))$ which enables us further, using the set S^0 and the functions $\rho_1^0(t, x)$ and $\rho_2^0(t, x)$ $(t, x) \in G$, to construct a set $S^1 \leqslant S^0$ and determine the functions $\rho_1^1(t,x)$ and $\rho_2^1(t,x)$, $(t,x) \in G$.

Suppose the position $(t, x) \in G$ is given. The H_1-solution and the H_2-solution (there cannot be one of them) correspond to this position as the initial position. We will denote by $w^1(\sigma)$, $t \leqslant \sigma \leqslant \vartheta$ and $w^2(\sigma)$, $t \leqslant \sigma \leqslant \vartheta$ the trajectories generated by the H_1-solution and H_2-solution respectively. We fix $\varepsilon > 0$ and put $\tau(t, \varepsilon) = \min\{t + \varepsilon, \vartheta\}$. Consider the vectors

$$s_1^0(t,x,\varepsilon)=w^1(\tau(t,\varepsilon))-x,\quad s_2^0(t,x,\varepsilon)=w^2(\tau(t,\varepsilon))-x \tag{3.2}$$

We define the vectors $u_1^0(t, x, \varepsilon)$, $v_1^0(t, x, \varepsilon)$, $u_2^0(t, x, \varepsilon)$ and $v_2^0(t, x, \varepsilon)$ from the conditions

$$\begin{aligned}
&\max_{u\in P,\, v\in Q} s_1^{0T}[f(t,x,u)+g(t,x,v)]=s_1^{0T}[f(t,x,u_1^0)+g(t,x,v_1^0)]\\
&\max_{u\in P,\, v\in Q} s_2^{0T}[f(t,x,u)+g(t,x,v)]=s_2^{0T}[f(t,x,u_2^0)+g(t,x,v_2^0)]
\end{aligned} \tag{3.3}$$

We now construct the auxiliary bimatrix 2×2 game (A, B) in which the first player has two strategies: "to choose u_1^0" and "to choose u_2^0". Similarly, the second player has two strategies: "to choose v_1^0" and "to choose v_2^0". The payoff matrices of the players are defined as follows:

$$A=\begin{Vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{Vmatrix},\quad B=\begin{Vmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{Vmatrix} \tag{3.4}$$

$$a_{ij}=\rho_1^0(\tau(t,\varepsilon),\, x+f(t,x,u_i^0,v_j^0)\tau(t,\varepsilon))$$

$$b_{ij}=\rho_2^0(\tau(t,\varepsilon),\, x+f(t,x,u_i^0,v_j^0)\tau(t,\varepsilon)),\quad i,j=1,2$$

It is obvious that $a_{11} \geqslant a_{21}$ and $b_{22} \geqslant b_{21}$, which enables us to eliminate situation (2, 1) from consideration when finding Nash equilibria in the game (A, B). It is easy to show that the bimatrix game (A, B) has at least one Nash equilibrium in pure strategies.

We choose as the controls $\tilde{u}(t, x, \varepsilon)$ and $\tilde{v}(t, x, \varepsilon)$, used by the players in position (t, x) for specified ε, the controls (u_i, v_j), which yield a Nash equilibrium. Four cases are possible.

1. Situation (1, 2) provides a unique Nash equilibrium. Then, the pair $u_1^0(t, x, \varepsilon)$, $\upsilon_2^0(t, x, \varepsilon)$ gives the controls of the players.
2. Situation (1, 1) provides a unique Nash equilibrium. Then, the pair $u_1^0(t, x, \varepsilon)$, $\upsilon_1^0(t, x, \varepsilon)$ gives the controls of the players.
3. Situation (2, 2) provides a unique Nash equilibrium. Then, the pair $u_2^0(t, x, \varepsilon)$, $\upsilon_2^0(t, x, \varepsilon)$ gives the controls of the players.
4. Situation (1, 1) and (2, 2) provides a unique Nash equilibrium. Then, the pair $u_1^0(t, x, \varepsilon)$, $\upsilon_1^0(t, x, \varepsilon)$ and $u_2^0(t, x, \varepsilon)$, $\upsilon_2^0(t, x, \varepsilon)$ with equal frequency.

Hence, the strategies $\tilde{u}(t, x, \varepsilon)$ and $\tilde{v}(t, x, \varepsilon)$ are defined for $(t, x) \in G$, $\varepsilon > 0$. In view of the non-uniqueness of the H_1-solution and the H_2-solution, the algorithm described specifies multivalued functions—strategies $\tilde{u}(t, x, \varepsilon)$ and $\tilde{v}(t, x, \varepsilon)$. Hence, when finding the motion we arrive at a differential equation with a multivalued right-hand side. As in [1, 2], we can construct approximation motions and limit motions of this equation.

We will denote by I^* the set of payoff vectors of the players (I_1, I_2) (1.3), reached on the set of limit motions of this equation. We will denote by S^1 the set of elements S^0 such that for each $s \in S^1$ we obtain $\xi \in I^*$, for which the inequality $s \geqslant \xi$ holds. Obviously, S^1 is compact. We will determine the functions $\rho^1_{1(t,x)}$ and $\rho^1_{2(t,x)}$, taking as $\rho^1_{i(t,x)}$ the value of the maximum payoff of the ith player in the set S^1.

Hence, the procedure $L(S^0, \rho_1^0(\cdot), \rho_2^0(\cdot))$ has enabled us to construct the set $S^1 \subseteq S^0$ and to determine the functions $\rho_1^1(t, x)$ and $\rho_2^1(t, x)$, $(t, x) \in G$.

Using the procedure $L(S^1, \rho_1^1(\cdot), \rho_2^1(\cdot))$ we obtain the set $S^2 \subseteq S^1$ and the functions $\rho_1^2(t, x)$ and $\rho_2^2(t, x)$, $(t, x) \in G$, etc. Taking the limit we obtain the non-empty set S^∞, which, in particular, may consist of a single point.

Definition 6. The set of strategies of the players in which the payoff of the players belongs to the set S^∞ will be called the solution of the NPDG-2 game.

As an illustration consider Example 1 of Section 2. The equation of motion and the constraints on the control are given in (2.6), while the performance figures of the players are given in (2.7).

Suppose $x(t_0) = 0$, $t_0 = 0$, $\vartheta = 2$, and the target points are $a^{(1)} = (5, 2)$, $a^{(2)} = (5, -2)$. The results of calculations show that the set S^∞ consists of a unique point (4, 0). The unique limit trajectory which leads to this set will be $x_1^*(t) = -t^2 + 4t$, $x^*_2(t) \equiv 0$, $0 \leqslant t \leqslant 2$. The resolving strategy has the structure (1.7), where $u^*(t) = (u_1^*(t), u_2^*(t))$, $\upsilon^*(t) = (\upsilon_1^*(t), \upsilon_2^*(t))$, $u_1^*(t) \equiv \upsilon_1^*(t) \equiv 1$, $u_2^*(t) \equiv \upsilon_2^*(t) \equiv 0$, $0 \leqslant t \leqslant 2$.

4. FORMALIZATION OF THE DIFFERENT TYPES OF BEHAVIOUR OF THE PLAYERS IN NPDG

When defining the solution concept we directed our attention to the natural, normal type of behaviour of the players, in which each player tries to maximize his or her own payoff functional. At the same time, in reality, other types of behaviour are often encountered, such as: altruism (what's better for the partner is better for me), animosity (what's worse for the partner is better for me). The extreme occurrences of these types of behaviour may be formalized within the model considered.

Definition 7. We will say that the first player is confined to the current position of altruistic (hostile) type of behaviour if his action in this position is directed exclusively towards maximizing (minimizing) the payoff of the second player I_2(1.3).

Definition 8. We will say that the first player is confined in the current position of paradoxical type of behaviour if his actions in this position are directed exclusively towards minimizing his own payoff I_1(1.3).

If the actions of the first player are directed towards maximizing his or her own payoff functional I_1 (as also specified in the initial formulation), we will say that the player is confined to a normal type of behaviour.

The second player is engaged in similar types of behaviour.

Note that a hostile type of behaviour of the players was in fact used above in the form of "punishment strategies", which occur in the structure of the solutions of game (1.7).

In the NPDG considered there are 16 different pairs of types of behaviour of the players, of which four correspond to the antagonistic interests of the players, four correspond to coinciding interests, and the remaining eight correspond to essentially non-antagonistic interests.

Note that the different types of behaviour of the players can be "mixed". Further, during the course of the game these "mixtures" may change depending on the information available to the players. It must, however, be emphasized that whatever types of behaviour the players maintain during the course of the game, the final payoffs of the players are "measured" with the quantities $\sigma_1(x(\vartheta))$ and $\sigma_2(x(\vartheta))$.

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